PROOF THAT THE DETERMINANT IS MULTIPLICATIVE

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https://www.patrickstevens.co.uk/misc/MultiplicativeDetProof/MultiplicativeDetProof.pdf

This is a very concrete proof of the multiplicity of the determinant. It contains no cleverness at all, and is simply manipulation of expressions.

Definition The *determinant* of a matrix A is given by

$$\sum_{\sigma \in S_n} \epsilon(\sigma) \prod_{i=1}^{n} A_{i,\sigma(i)}$$

where S_n is the symmetric group on n elements, and ϵ is the signature of that element.

Lemma 0.1. *Let $\rho \in S_n$, and let A be a matrix. Then*

$$\sum_{\sigma \in S_n} \epsilon(\sigma) \prod_{i=1}^{n} A_{\rho(i)\sigma(i)} = \epsilon(\rho) \det(A)$$

Proof.

$$\begin{aligned}
\epsilon(\rho) \det(A) &= \epsilon(\rho) \sum_{\sigma \in S_n} \epsilon(\sigma) \prod_{i=1}^{n} A_{i,\sigma(i)} \\
&= \sum_{\sigma \in S_n} \epsilon(\sigma\rho) \prod_{i=1}^{n} A_{i,\sigma(i)} \\
&= \sum_{\sigma \in S_n} \epsilon(\sigma\rho) \prod_{i=1}^{n} A_{\rho(i),\sigma(\rho(i))} \\
&= \sum_{\tau \in S_n} \epsilon(\tau) \prod_{i=1}^{n} A_{\rho(i),\tau(i)} \\
&= \sum_{\tau \in S_n} \epsilon(\sigma) \prod_{i=1}^{n} A_{\rho(i),\sigma(i)}
\end{aligned}$$

□

Theorem 0.2.

$$\det(AB) = \det(A)\det(B)$$

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Proof. We use summation convention throughout.

$$\begin{aligned}\det(AB) &= \sum_{\sigma \in S_n} \epsilon(\sigma) \prod_{i=1}^{n} (AB)_{i,\sigma(i)} \\ &= \sum_{\sigma \in S_n} \epsilon(\sigma) A_{1,k_1} B_{k_1,\sigma(1)} A_{2,k_2} B_{k_2,\sigma(2)} \dots A_{n,k_n} B_{k_n,\sigma(n)} \\ &= \sum_{\sigma \in S_n} \epsilon(\sigma) A_{1,k_1} A_{2,k_2} \dots A_{n,k_n} B_{k_1,\sigma(1)} \dots B_{k_n,\sigma(n)}\end{aligned}$$

But the $k_1, \dots, k_n$ only ever contribute when they are a permutation of $1, \dots, n$, because (assuming wlog $k_1 = k_2$) for each σ^+ there exists σ^- such that $\sigma^+(1) = \sigma^-(2), \sigma^-(1) = \sigma^+(2), \sigma^-(k) = \sigma^+(k)$ for other k. Then we have

$$A_{1,k_1} B_{k_1,\sigma^+(1)} A_{2k_1} B_{k_1\sigma^+(2)} \text{terms} = A_{1,k_1} B_{k_1,\sigma^-(1)} A_{2k_1} B_{k_1\sigma^-(2)} \text{terms}$$

and because ϵ negates the sign, we have that these two terms cancel.

Hence the sum over k_i is in fact a sum over all ρ such that $\rho(i) = k_i$ for all i: Then

$$\det(AB) = \sum_{\rho \in S_n} \sum_{\sigma \in S_n} \epsilon(\sigma) A_{1,\rho(1)} A_{2,\rho(2)} \dots A_{n,\rho(n)} B_{\rho(1),\sigma(1)} \dots B_{\rho(n),\sigma(n)}$$

Applying the lemma gives

$$\begin{aligned}\det(AB) &= \det(B) \sum_{\rho \in S_n} \epsilon(\rho) A_{1,\rho(1)} A_{2,\rho(2)} \dots A_{n,\rho(n)} \\ &= \det(B) \det(A)\end{aligned}$$

□