# TENNENBAUM'S THEOREM

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https://www.patrickstevens.co.uk/misc/Tennenbaum/Tennenbaum.pdf

## 1. Introduction

**Theorem 1.1** (Tennenbaum's Theorem). *Let $\mathfrak{M}$ be a countable non-standard model of Peano arithmetic, whose carrier set is $\mathbb{N}$. Then it is not the case that $+$ and $\times$ have decidable graphs in the model.*

*Notation.* We will use the notation $\{e\}$ to represent the $e$th Turing machine. $e$ is considered only to be a standard integer here. For example, we might view the Gödel numbering scheme as being "convert from ASCII and then interpret as a Python program".

*Remark.* How might our standard Turing machine refer to a nonstandard integer? The ground set of our nonstandard model is $\mathbb{N}$: every nonstandard integer has a standard one which represents it in $\mathbb{N}$. Perhaps $4 \in \mathbb{N}$ is the object that the nonstandard model $\mathfrak{M}$ thinks is the number 7, for instance. So the way a Turing machine would refer to the number 7-in-the-model is to use 4 in its source code.

What does it mean for $+$ to have a decidable graph? Simply that there is some (standard) natural $n$ such that, when we unpack $n$ into instructions for running a Turing machine, we obtain a machine that takes three naturals (that is, standard naturals) $a, b, c$ and outputs 1 iff, when we take the referents $a', b', c'$ of $a, b, c$ in the model $\mathfrak{M}$, it is true that $a' +_{\mathfrak{M}} b' = c'$.

*Example.* A strictly standard-length program may halt in nonstandard time, when interpreted in a nonstandard model. Indeed, fix some nonstandard "infinite" $n$ (i.e. $n$ is not a standard natural). Then the following program halts after $n$ steps.

```
ans = 0;
for i = 1 to n:
  ans := ans + 1;
end
HALT with output ans;
```

## 2. Overview of the proof

The proof *est omnis divisa in partes tres.*

(1) In any model, there is some pair of semidecidable but recursively inseparable sets.

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(2) We can use these to create an undecidable set of true standard naturals which can, in some sense, be coded up into a (nonstandard) natural in our model.
(3) If $+$ and $\times$ were decidable, then the coding process would produce an object which would let us decide the undecidable set; contradiction.

## 3. Existence of recursively inseparable sets

This is fairly easy. Take $A = \{e : \{e\}(e) \downarrow= 0\}$ and $B = \{e : \{e\}(e) \downarrow> 0\}$, where $\downarrow=$ means "halts and is equal to", and $\downarrow>$ means "halts and is greater than". Recall that $e$ must be standard.

Now, suppose there were a (standard) integer $n$ such that $\{n\}$ were the indicator function on set $X$, where $X \cap B = \emptyset$ and $A \subseteq X$. Then what is $\{n\}(n)$? If it were 0, then $n$ is not in $X$, so $n$ is not in $A$ and so $\{n\}(n)$ doesn't halt at 0. That's a contradiction. If it were 1, then $n$ is in $X$ and hence is not in $B$, so $\{n\}(n)$ doesn't halt at something bigger than 0; again a contradiction.

So we have produced a pair of sets which are both semidecidable but are recursively inseparable, in the sense that no standard integer $n$ has $\{n\}$ deciding a superset $X$ of $A$ where $X \cap B = \emptyset$. (This is independent of the model of PA we were considering; it's purely happening over the ground set.)

## 4. Coding sets of naturals as naturals

We can take any set of (possibly nonstandard) naturals and code it as a (possibly nonstandard) natural, as follows. Given $\{n_i : i \in I\}$, code it as $\sum_{i \in I} 2^{n_i}$. If $+$ and $\times$ are decidable, then this is a decidable coding scheme. (The preceding line is going to be where our contradiction arises, right at the end of the proof!)

Notice that if $I$ is "standard-infinite" (that is, it contains nonstandardly-many elements) then the resulting code is nonstandard. Additionally if any $n_i$ is strictly-nonstandard.

## 5. Undecidable set in $\mathfrak{M}$

Take our pair of recursively inseparable semidecidable sets: $\mathfrak{A}$ and $\mathfrak{B}$. (We constructed them explicitly earlier, but now we don't care what they are.) Recalling a theorem that being semidecidable is equivalent to being a projection of a decidable set, write $A$ for a decidable set such that $(\exists y)[(n, y) \in A]$ if and only if $n \in \mathfrak{A}$, and similarly for $B$. (The quantifiers range over $\mathbb{N}$, because $A$ and $B$ consist only of standard naturals, being subsets of the ground set.)

By their recursive-inseparability, they are in particular disjoint, so we have

$$(\forall n)[(\exists x)(\langle n, x \rangle \in A) \to \neg(\exists y)(\langle n, y \rangle \in B)]$$

where the quantifiers all range over $\mathbb{N}$. Equivalently,

$$(\forall n)(\forall x)(\forall y)(\neg \langle n, x \rangle \in A \vee \neg \langle n, y \rangle \in B)$$

If we bound the quantifiers by any standard $m = SS \ldots S(0)$ (which we explicitly write out, so it's absolute between all models of PA), we obtain an expression which our nonstandard model believes, because the expression is absolute for PA:

$$(\forall n < m)(\forall x < m)(\forall y < m)(\neg \langle n, x \rangle \in A \vee \neg \langle n, y \rangle \in B)$$

This is true for every standard $m$, and so it must be true for some nonstandard $m$ by overspill, since $\mathfrak{M}$ doesn't know how to distinguish between standard and nonstandard elements. If the property were only ever true for standard $m$, then $\mathfrak{M}$ could identify nonstandard $m$ by checking whether that property held for $m$.

Let $e$ be strictly nonstandard such that

$$\mathfrak{M} \vDash (\forall n < e)(\forall x < e)(\forall y < e)(\langle n, x\rangle \notin A \vee \langle n, y\rangle \notin B) \tag{1}$$

where we note that this time $e$ is not written out explicitly as $SS \dots S(0)$ because it's too big to do that with.

Finally, we define our undecidable set $X \subseteq \mathbb{N}$ of *standard* naturals to be those standard naturals $x$ such that

$$\mathfrak{M} \vDash (\exists y < e)(\langle x, y\rangle \in A)$$

This is undecidable in the standard sense: there are no standard $m$ such that $\{m\}$ is the indicator function of $X$. Indeed, I claim that $X$ separates $\mathfrak{A}$ and $\mathfrak{B}$. (Recall that all members of $X$, $\mathfrak{A}$ and $\mathfrak{B}$ are standard.)

- If $a \in \mathfrak{A}$ then there is some standard natural $n$ such that $\langle a, n\rangle \in A$; and $n$ is certainly less than the nonstandard $e$. Hence $a \in X$.
- If $b \in \mathfrak{B}$, then there is standard $n$ such that $\langle b, n\rangle \in B$. Then $n < e$, so by (1) we have $\langle b, x\rangle \notin A$ for all $x < e$. That is, $b \notin X$.

## 6. Coding up $X$

Now if we code up $X$, which is undecidable, using our coding scheme

$$\{n_i : i \in I\} \mapsto \sum_{i \in I} 2^{n_i}$$

we obtain some nonstandard natural; say $p = \sum_{x \in X} 2^x$. Supposing the $+$ and $\times$ relations to be decidable, this coding is decidable. Remember that $X$ is a set of standard naturals which is undecidable: no standard Turing machine decides $X$.

But here is a procedure to determine whether a standard element $i \in \mathbb{N}$ is in $X$ or not:

(1) Take the $i$th bit of $p$. (This is decidable because $+$ and $\times$ are.)
(2) Return "not in $X$" if the $i$th bit is 0.
(3) Otherwise return "is in $X$".

This contradicts the undecidability of $X$.

## 7. Acknowledgements

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