TOKYO 2016 GRADUATE SCHOOL ENTRANCE EXAM

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https://www.patrickstevens.co.uk/misc/TokyoEntrance2016/TokyoEntrance2016.pdf

1. Question 2

1.1. Prove that $S = 2\pi \int_{-1}^{1} F(y, y') \, \mathrm{d}x$. The surface may be parametrised as

$$S(x, \theta) = (x, y(x)\cos(\theta), y(x)\sin(\theta))$$

where $\theta \in [0, 2\pi)$ and $x \in [-1, 1]$.

Hence

$$\frac{\partial S}{\partial x} = (1, y'(x)\cos(\theta), y'(x)\sin(\theta))$$

and

$$\frac{\partial S}{\partial \theta} = (0, -y(x)\sin(\theta), y(x)\cos(\theta))$$

so the surface element

$$\mathrm{d}\Sigma = \left|(1, y'(x)\cos(\theta), y'(x)\sin(\theta)) \times (0, -y(x)\sin(\theta), y(x)\cos(\theta))\right| \mathrm{d}x\mathrm{d}\theta$$

i.e.

$$y\sqrt{1 + (y')^2}$$

The integral is therefore

$$\int_0^{2\pi} \int_{-1}^{1} y\sqrt{1 + (y')^2}\mathrm{d}x\mathrm{d}\theta$$

as required.

1.2. Prove the first integral of the Euler-Lagrange equation. We know the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'}$$

Now,

$$\frac{\mathrm{d}F}{\mathrm{d}x} = \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial F}{\partial y'}\frac{\mathrm{d}y'}{\mathrm{d}x}$$

so substituting Euler-Lagrange into this:

$$\frac{\mathrm{d}F}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial F}{\partial y'}\frac{\mathrm{d}y'}{\mathrm{d}x}$$

Notice the right-hand side is just what we get by applying the product rule: it is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

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The result follows now by simply integrating both sides with respect to x.

1.3. **Solve the differential equation.** Just substitute $F(y, y') = y\sqrt{1+(y')^2}$:

$$y\sqrt{1+(y')^2} - y'\left[\frac{1}{2}y(1+(y')^2)^{-1/2} \cdot 2y'\right] = c$$

which can be simplified to

$$y(1+(y')^2)^{-1/2}\left[(1+(y')^2) - (y')^2\right] = c$$

i.e.

$$y^2 - c^2 = (cy')^2$$

If $c = 0$ then this is trivial: $y = 0$. From now on, assume $c \neq 0$; then since y is known to be positive, $c > 0$.

Invert:

$$\frac{c^2}{y^2 - c^2} = \left(\frac{dx}{dy}\right)^2$$

so

$$\frac{dx}{dy} = \pm\frac{c}{\sqrt{y^2 - c^2}}$$

which is a standard integral:

$$x = \pm c\log(y + \sqrt{y^2 - c^2}) + K$$

Also $y(-1) = 2 = y(1)$, so

$$\{1, -1\} = \{c\log(2 + \sqrt{4 - c^2}) + K, -c\log(2 + \sqrt{4 - c^2}) + K\}$$

which means $K = 0$.

Then

$$\exp\left(\pm\frac{x}{c}\right) = y + \sqrt{y^2 - c^2}$$

Since $y(1) = 2$, we have

$$\exp(\pm 1/c) = 2 + \sqrt{4 - c^2}$$

and in particular (since $c > 0$) we have the $\pm$ on the left-hand side being positive; that is the expression c is required to satisfy.

Rearrange:

$$y = \frac{c + \exp(2\frac{x}{c})}{2\exp\left(\frac{x}{c}\right)}$$

which completes the question.

2. Question 3

2.1. **Part 1.** We must put one ball into each box. Then we are distributing $n - r$ balls freely among r boxes, so the answer is

$$\binom{n - r - 1}{r - 1}$$

(standard stars-and-bars result).

2.2. **Part 2.** Consider the n black balls laid out in a line; we are interspersing the m white balls among them. Equivalently, we have $n+1$ boxes (represented by the gaps between black balls) and we are trying to put m balls into them. By stars-and-bars again, the answer is $\binom{n}{m-1}$.

2.3. **Part 3.** Condition on the colour of the first ball, and write l for the length of the first run. Then

$$P_{n,m}(r,s) = \frac{n}{n+m} \sum_{l=0}^{n} P_{n-l,m}(r-1,s) + \frac{m}{n+m} \sum_{l=0}^{m} P_{n,m-l}(r,s-1)$$

Also $P_{n,m}(0,s) = \chi[n=0]\chi[s=1]$ where χ is the indicator function, and $P_{n,m}(r,0) = \chi[m=0]\chi[r=1]$.

2.4. **Part 4.**

2.5. **Part 5.** If $m \leq n$, then the sum is

$$\sum_{l=0}^{m} \binom{n}{l} \binom{m}{m-l}$$

which is the x^m coefficient of the left-hand side and hence of the right-hand side.

If $m > n$, then the sum is

$$\sum_{l=0}^{n} \binom{n}{n-l} \binom{m}{l}$$

which is the x^n coefficient of the left-hand side and hence of the right-hand side.

For the second equation: this follows by setting $n \mapsto n-1$ in the above.

2.6. **Part 6.**